Two-Sided Sub-Diffusivity in the Dynamical Discrete Web

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Abstract

The dynamical discrete web (DyDW) is a system of one-dimensional coalescing random walks that evolves in an extra dynamical time parameter, τ. At any deterministic τ the paths behave as coalescing simple symmetric random walks. It was shown in 2009 by Fontes, Newman, Ravishankar and Schertzer that there exist exceptional times at which the path starting at the origin violates the law of the iterated logarithm. To be specific, they show that there exist exceptional dynamical times, τ, at which the path from the origin, S_0^τ, is K-subdiffusive, meaning $S_0^\tau(t) \leq j + K\sqrt{t}$ for all t, where t is the random walk time, and j is some constant. The first goal of this paper is to establish the existence of exceptional times for which the path from the origin is K-subdiffusive in both directions, i.e. τ such that $|S_0^\tau(t)| \leq j + K\sqrt{t}$ for all t. We then obtain upper and lower bounds for the Hausdorff dimensions of this set, and related sets, of exceptional times.

1 Introduction

This paper examines the dynamical discrete web (DyDW), a system of coalescing random walks that evolves in a continuous dynamical time parameter. The dynamical discrete web was introduced by Howitt and Warren in [7]. The DyDW and related systems have been considered as models for erosion and other hydrological phenomena (see [9],[1]). We examine "exceptional times" for the DyDW. These are dynamical times at which paths from the DyDW display behavior that would have probability zero for a standard random walk, or for the DyDW observed at a deterministic time.

Now we define the dynamical discrete web, and briefly describe our main result. This paper follows [2] closely; for a more thorough introduction to the subject see Section 1 of their paper.

To discuss the DyDW, we first define the discrete web (DW). The discrete web is a system of coalescing one-dimensional simple symmetric random walks. To construct it, we independently assign to each point in $\mathbb{Z}^2_{\text{even}} := \{(x,t) \in \mathbb{Z}^2 : \ x+t \text{ is even}\}$ a symmetric, ±1-valued Bernoulli random variable, $\xi_{(x,t)}$. We then draw an arrow from (x,t) to $(x+\xi_{(x,t)}, t+1)$ (see Figure 1). For each $(x,t) \in \mathbb{Z}^2_{\text{even}}$, we let $S_{(x,t)}(t)$ be the path that starts at (x,t) and follows the arrows from there. The discrete web is the collection of all such paths for $(x,t) \in \mathbb{Z}^2_{\text{even}}$. As the figures and the ordering of (x,t) suggest, we let the path time coordinate, t, run vertically, and the space coordinate, x, run horizontally. Future references to left/right, vertical/horizontal should be understood according to this convention.

The DyDW was first introduced by Howitt and Warren in [7]. It is a discrete web that evolves in an extra dynamical time parameter, τ, by letting the arrows independently switch directions as τ increases. To accomplish this, we assign to each $(x,t) \in \mathbb{Z}^2_{\text{even}}$ an independent, rate one Poisson clock. When the clock at (x,t) rings, we reset the arrow at (x,t) by replacing it with a new, independent arrow (that may or may not agree with the previous arrow). This corresponds to replacing the $\xi_{(x,t)}$ from the DW, with right-continuous τ-varying versions, $\xi^\tau_{(x,t)}$. We then let

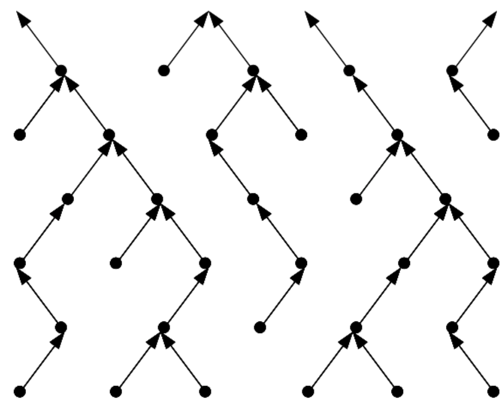

Figure 1: A partial realization of the discrete web. Each arrow independently points left or right with probability 1/2. In the dynamical discrete web, each arrow has an independent Poisson clock and resets whenever it rings.

$\mathcal{W}(\tau)$ denote the discrete web constructed from the $\xi^\tau_{(x,t)}$'s, and let $S^\tau_{(x,t)}(t)$ denote the path from $\mathcal{W}(\tau)$ starting at (x,t). Note that at any deterministic τ, $\mathcal{W}(\tau)$ is distributed as a discrete web.

Exceptional times for the DyDW were first studied by Fontes, Newman, Ravishankar and Schertzer in [2]. There they show that there exist exceptional times for the DyDW at which S^τ_0 is subdiffusive in one direction. That is, for sufficiently large K, j:

$$\mathbb{P}(\exists \tau \in [0,1] \text{ s.t. } S^\tau_0(t) \leq j + K\sqrt{t} \text{ for all } t \geq 0) > 0. \tag{1}$$

Their work was motivated by the study of exceptional times for dynamical percolation, see [6], [10]. Similarly to the DyDW, dynamical percolation consists of a lattice of Bernoulli random variables which reset according to independent Poisson processes. For static (non-dynamical) percolation with critical edge probabilities it is believed that no infinite cluster should exist. This is proven for dimension two and large dimensions (see [5], for example). In [10] it was shown that critical two-dimensional dynamical percolation has exceptional times where this fails, i.e. where an infinite cluster exists. However, no such exceptional times exist for large dimensions, see [6].

In [2] they use techniques similar to those used for dynamical percolation to prove (1) and examine the dimensions of the corresponding sets of exceptional τ. We extend their arguments to show the existence of exceptional times for the dynamical discrete web at which S^τ_0 is subdiffusive in both directions. To be specific, we prove:

Theorem 1. *For K, j sufficiently large:*

$$\mathbb{P}(\exists \tau \in [0,1] \textit{ s.t. } |S^\tau_0(t)| \leq j + K\sqrt{t} \textit{ for all } t \geq 0) > 0. \tag{2}$$

An immediate consequence of this is:

Corollary 1. *For K sufficiently large:*

$$\mathbb{P}(\exists \tau \in [0,1] \textit{ s.t. } \limsup |S^\tau_0(t)/\sqrt{t}| \leq K) > 0. \tag{3}$$

In the final section of the paper we examine the sets of exceptional times and study their Hausdorff dimensions. That is, we look at the sets:

$$\{\tau \in [0, \infty) : \exists j \text{ s.t. } |S_0^\tau(t)| \leq j + K\sqrt{t} \text{ for all } t \geq 0\}, \tag{4}$$

$$\{\tau \in [0, \infty) : \limsup |S_0^\tau(t)/\sqrt{t}| \leq K\}, \tag{5}$$

and derive upper and lower bounds for their Hausdorff dimensions, as functions of K. Our bounds are analogous to, and motivated by, those from [2] for the one-sided case. As in the one-sided case, the dimensions tend to 1 as K goes to ∞. For small K it is known that (4) is empty, see Proposition 5.8 of [2]. This implies (5) is also empty for small K, see Section 5. Our analysis of (5) is helped by noting that (5) only depends on arrows with arbitrarily large time coordinate (almost surely). This means (5) can be analysed using tail events, allowing us to improve the lower bound slightly relative to the methods of [2]. The two sets (4) and (5) have the same dimensions, except for at most countably many values of K (see Section 5 for details). It would seem natural that the two dimensions would be the same for all K, but we are unable to prove this.

2 Structure of the Proof of Theorem 1

As in [2], we show that subdiffusivity occurs by showing that a series of "rectangle events" occur. First, we define our rectangles. Let $\gamma > 1$ and $d_k = 2(\lfloor \frac{\gamma^k}{2} \rfloor + 1)$. Let R_0 be the rectangle with vertices $(-d_0, 0)$, $(+d_0, 0)$, $(-d_0, d_0^2)$ and $(+d_0, d_0^2)$. Given R_k we take R_{k+1} to be the rectangle of width $2d_{k+1}$ and height d_{k+1}^2, that is centered about the y-axis, and stacked on top of R_k (see Figure 2). An easy computation shows that the entire stack of rectangles lies between the graphs of $-j - K\sqrt{t}$ and $j + K\sqrt{t}$, where j, K depend on γ. For example, we can take $j = 2, K = \gamma$, see Proposition 2 of Section 5. Thus if S_0^τ stays within the stack, it will be subdiffusive in both directions.

Remark 1. Notice that this gives a bound with left-right symmetry. If we wish to study exceptional times where $-j_L - K_L\sqrt{t} \leq S_0^\tau(t) \leq j_R + K_R\sqrt{t}$, we can skew our rectangles. This can be accomplished by horizontally scaling the left and right halves of each rectangle by C_L and C_R, respectively (and rounding out to the nearest point in $\mathbb{Z}^2_{even}$). For the sake of simplicity of our arguments (and notation) we will largely ignore the asymmetrical case. However, it should be noted that our results easily extend to the asymmetrical case, using the above construction.

Let t_k denote the time coordinate of the lower edge of R_k. For $k \geq 1$, let l_k denote the upper left vertex of R_{k-1} and r_k the upper right vertex of R_{k-1}. We would like to define our rectangle events, B_k^τ, as:

$$\begin{aligned} B_0^\tau :=& \{|S_0^\tau(t)| \leq d_o \;\; \forall t \in [0, t_1]\}, \\ B_k^\tau :=& \{|S_{l_k}^\tau(t)| \leq d_k \text{ and } |S_{r_k}^\tau(t)| \leq d_k \;\; \forall t \in [t_k, t_{k+1}]\} \;\text{ for } k \geq 1. \end{aligned}$$

Then on the event $\bigcap_{k\geq 0} B_k^\tau$, S_0^τ will stay in the stack of rectangles, and thus be subdiffusive in both directions. This follows from the discussion above, combined with the fact that paths in the discrete web do not cross. Thus if for some γ we can show:

$$\mathbb{P}(\exists \tau \in [0,1] \text{ s.t. } \bigcap_{k\geq 0} B_k^\tau(\gamma) \text{ occurs}) > 0, \tag{6}$$

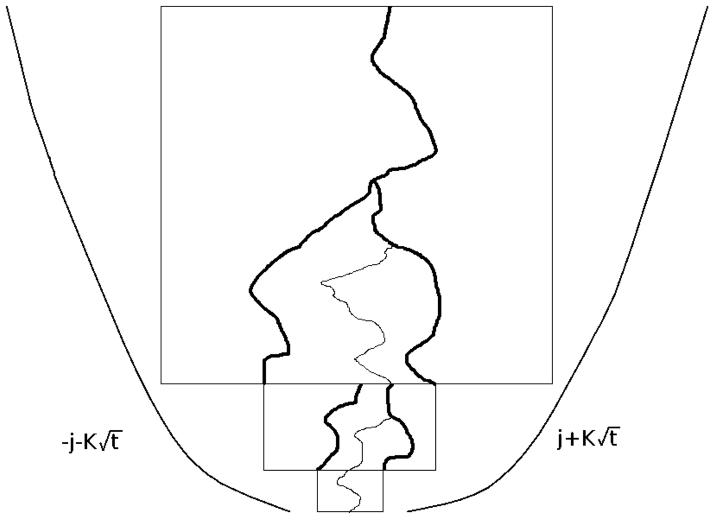

Figure 2: Rough sketch of the first three rectangles and paths for which the B_k's occur. The darker paths are the $S^\tau_{l_k}$'s and $S^\tau_{r_k}$'s. The lighter path is S^τ_0.

then Theorem 1 will follow immediately.

To prove (6), we will need to understand the interaction between pairs of paths from the DyDW. This can be described as a combination of coalescing (if the paths have the same dynamical time) and sticking (if the dynamical times differ). Let S^τ_z be the path from $z = (x,t) \in \mathbb{Z}^2_{\text{even}}$ at dynamical time τ, and let $S^{\tau'}_{z'}$ be the path from $z' = (x', t')$ at dynamical time τ'. The paths will evolve independently until they meet at some time $t^* \geq \text{Max}(t,t')$. If $\tau = \tau'$, the paths coalesce when they meet, otherwise they "stick". To be precise, let $x^* := S^\tau_z(t^*) = S^{\tau'}_{z'}(t^*)$ and let $z^* = (x^*, t^*)(\in \mathbb{Z}^2_{\text{even}})$. Then if the clock at z^* has not rung in $[\tau, \tau']$ (WLOG assume $\tau < \tau'$), the two paths will follow the same arrow on $[t^*, t^*+1]$. We will say the paths are sticking on $[t^*, t^*+1]$. The paths continue to stick until they reach a site whose clock has rung, at which point they follow independent arrows. Note that these independent arrows may agree, but this will not be considered sticking.

To prove Theorem 1, we would like to show (6). Unfortunately, we are not able to prove (6) directly. The problem arises in the interaction between sticking and coalescing (to be specific, (12)-(14) fail for B^τ_k, so we are unable to establish (16)). To get around this, we construct a larger system where the relevant paths do no not coalesce. In addition to the main DyDW, $\mathcal{W}(\tau)$, we will need an independent, secondary DyDW, $\hat{\mathcal{W}}(\tau)$. From now on, all "arrows", "clock rings", etc. should be understood to refer to $\mathcal{W}(\tau)$ (the main DyDW), unless otherwise specified.

Given $S^\tau_{l_k}$ and $S^\tau_{r_k}$ we want to construct non-coalescing versions, $X^\tau_{l_k}$ and $X^\tau_{r_k}$. We accomplish this by letting $X^\tau_{l_k} = S^\tau_{l_k}$, and taking $X^\tau_{r_k}$ to be the path from r_k that follows the arrows (from $\mathcal{W}(\tau)$) unless it meets $X^\tau_{l_k}$. If $X^\tau_{r_k}$ meets $X^\tau_{l_k}$ at space-time $z^* = (x^*, t^*) \in \mathbb{Z}^2_{\text{even}}$, then on $[t^*, t^*+1]$ we let $X^\tau_{r_k}$ follow the arrow at z^* from $\hat{\mathcal{W}}(\tau)$ (at dynamical time τ). At time t^*+1 we repeat this, following $\hat{\mathcal{W}}(\tau)$ if the paths are together, but following $\mathcal{W}(\tau)$ otherwise. Continuing in this manner we get an independent pair of non-coalescing simple symmetric random walks $X^\tau_{l_k}$ and $X^\tau_{r_k}$. Now

we define new rectangle events, C_k^τ:

$$\begin{aligned} C_0^\tau :=& B_0^\tau, \\ C_k^\tau :=& \{|X_{l_k}^\tau(t)| \le d_k \text{ and } |X_{r_k}^\tau(t)| \le d_k \ \ \forall t \in [t_k, t_{k+1}]\} \ \text{ for } k \ge 1. \end{aligned}$$

Notice that C_k^τ implies B_k^τ. This is because the only difference between $X_{l_k}^\tau$,$X_{r_k}^\tau$ and $S_{l_k}^\tau$,$S_{r_k}^\tau$ is the (possible) extension of $X_{r_k}^\tau$ beyond the initial meeting point. So if we can show:

$$\mathbb{P}(\exists \tau \in [0,1] \text{ s.t. } \bigcap_{k\ge 0} C_k^\tau \text{ occurs}) > 0, \tag{7}$$

then (6), and thus Theorem 1, will follow immediately. The next two sections will be devoted to proving (7).

3 A Decorrelation Bound

Throughout this section we assume $\tau, \tau' \in [0,1], \tau < \tau'$ and we fix arbitrary $k \ge 1, \gamma > 1$. We also translate the paths to start at $t = 0$. That is, we set $Y_l^\tau(t) := X_{l_k}^\tau(t_k + t)$ and $Y_r^\tau(t) := X_{r_k}^\tau(t_k + t)$ (k is fixed so we drop it from the notation). We will also consider diffusively rescaled versions of these paths, $\tilde{Y}_l^\tau(t) := Y_l^\tau(td_k^2)/d_k$ and $\tilde{Y}_r^\tau(t) := Y_r^\tau(td_k^2)/d_k$. The relevant "rectangle event" is then:

$$\begin{aligned} C^\tau :=& \{|Y_l^\tau(t)| \le d_k \text{ and } |Y_r^\tau(t)| \le d_k \ \ \forall t \in [0, d_k^2]\} \\ =& \{|\tilde{Y}_l^\tau(t)| \le 1 \text{ and } |\tilde{Y}_r^\tau(t)| \le 1 \ \ \forall t \in [0,1]\}. \end{aligned}$$

Similarly to [2] we define $\Delta := \dfrac{1}{d_k|\tau - \tau'|}$ (take their $\delta = d_k^{-1}$). As in [2], the key ingredient for the proof of (7) is a decorrelation bound for the rectangle events:

Proposition 1. *There exist $c, a \in (0,\infty)$ such that:*

$$\mathbb{P}(C^\tau \cap C^{\tau'}) \le \mathbb{P}(C^0)^2 + c\,(\Delta)^a \le \mathbb{P}(C^0)^2 + c\left(\frac{1}{\gamma^k|\tau - \tau'|}\right)^a,$$

with a, c independent of k, τ and τ'.

Note that the second inequality is follows immediately from the definitions of Δ, d_k. The remainder of this section is devoted to proving the first inequality, and thus Proposition 1. The structure is similar to the proof of Lemma 3.1 from [2], with a few necessary modifications.

As discussed in the previous section, paths from the DyDW at different dynamical times interact by sticking. This sticking leads to dependence between the web paths. Our modified paths (the Y_τ's) have their own version of sticking that is slightly more complicated. To prove Proposition 1 we will prove bounds for the amount of sticking, which will allow us to bound the dependence between the C^τ's. We begin with some notation and definitions.

We call $n \in \mathbb{Z}$ a "sticking time" if a Y^τ-path and a $Y^{\tau'}$-path are at the same spot and follow the same arrow at time n. This can happen in five ways:

(i) $Y_l^\tau(n) = Y_l^{\tau'}(n)$ no ring in $[\tau, \tau']$,

(ii) $Y_l^\tau(n) = Y_r^{\tau'}(n) \ne Y_l^{\tau'}(n)$ no ring in $[\tau, \tau']$,

(iii) $Y_l^{\tau'}(n) = Y_r^{\tau}(n) \neq Y_l^{\tau}(n)$ no ring in $[\tau, \tau']$,

(iv) $Y_r^{\tau}(n) = Y_r^{\tau'}(n) \neq Y_l^{\tau}(n), Y_l^{\tau'}(n)$ no ring in $[\tau, \tau']$,

(v) $Y_r^{\tau}(n) = Y_r^{\tau'}(n) = Y_l^{\tau}(n) = Y_l^{\tau'}(n)$ no $\hat{\mathcal{W}}$-ring in $[\tau, \tau']$.

We will call (i) an ll(left-left)-sticking time, (ii) an lr-sticking time, (iii) an rl-sticking time, and (iv),(v) will both be rr-sticking times. These names refer to which pair(s) of paths are sticking at time n.

Given $s \in [0, \infty)$ let n_s be the unique $n \in \mathbb{Z}$ such that $s \in [n, n+1)$. We define:

$$g(s) := \begin{cases} 0 & \text{if } n_s \text{ is a sticking time} \\ 1 & \text{otherwise} \end{cases}$$

$$G(t) := \int_0^t g(s) ds.$$

We will also need:

$$g_{ll}(s) := \begin{cases} 0 & \text{if } n_s \text{ is an } ll\text{-sticking time} \\ 1 & \text{otherwise} \end{cases}$$

$$G_{ll}(t) := \int_0^t g_{ll}(s) ds,$$

and G_{lr}, G_{rl}, G_{rr}, which are defined analogously.

Notice that $t - G(t)$ is the amount of time spent sticking up to time t. So if we make the time change $t \to t - G(t)$ we will include only the sticking steps. Similarly if we make the time change $t \to G(t)$ we will include only the non-sticking steps. This allows us to decompose the paths as:

$$\begin{aligned} Y_l^{\tau}(t) =& Y_{l_d}^{\tau}(G(t)) + Y_{l_s}^{\tau}(t - G(t)), \\ Y_r^{\tau}(t) =& Y_{r_d}^{\tau}(G(t)) + Y_{r_s}^{\tau}(t - G(t)), \\ Y_l^{\tau'}(t) =& Y_{l_d}^{\tau'}(G(t)) + Y_{l_s}^{\tau'}(t - G(t)), \\ Y_r^{\tau'}(t) =& Y_{r_d}^{\tau'}(G(t)) + Y_{r_s}^{\tau'}(t - G(t)), \end{aligned} \tag{8}$$

with $Y_{l_d}^{\tau}(0) = Y_l^{\tau}(0) = -d_{k-1}$, $Y_{r_d}^{\tau}(0) = Y_r^{\tau}(0) = d_{k-1}$, and $Y_{l_s}^{\tau}(0) = Y_{r_s}^{\tau}(0) = 0$ (similarly for τ'). Recall that the Y_{l_d}'s and Y_{r_d}'s include only the non-sticking steps of each walk. This means that the τ-paths and the τ'-paths follow different, independent arrows, and thus are independent.

To make the above splitting work for the $\tilde{Y}$'s the appropriate rescaling of G is $\bar{G}(t) := G(td_k^2)/d_k^2$. We then make the time changes $t \to t - \bar{G}(t)$ and $t \to \bar{G}(t)$. We would like a bound for $t - \bar{G}(t)$, the amount of sticking for the rescaled paths in $[0, t]$. This is given by the following adaptation of Lemma 3.4 from [2]:

Lemma 1. *For any* $0 < \beta < 1$

$$\mathbb{P}\left(\sup_{t \in [0,1]} (t - \bar{G}(t)) \geq \Delta^{\beta}\right) \leq c'' \Delta^{1-\beta},$$

where $c'' \in (0, \infty)$ *is independent of* k, τ *and* τ'.

Proof. Notice that by definition:

$$t - G(t) \leq \underset{(a)}{(t - G_{ll}(t))} + \underset{(b)}{(t - G_{lr}(t))} + \underset{(c)}{(t - G_{rl}(t))} + \underset{(d)}{(t - G_{rr}(t))}. \tag{9}$$

Let $C(t)$ be defined as in [2], i.e. such that $t-C(t)$ is the sticking time for S_0^τ and $S_0^{\tau'}$. We claim that each of $(a),(b),(c),(d)$ is stochastically bounded by $t-C(t)$ (given random variables X, Y, X is said to stochastically bound Y if $\mathbb{P}(Y > x) \leq \mathbb{P}(X > x)$ for all $x \in \mathbb{R}$). For (a) this is obvious, since $t - G_{ll}(t) \overset{d}{=} t - C(t)$ (equal in distribution). This is because the Y_l's are just translated web paths and the DyDW is invariant under space-time translations. We now concentrate on (d); (b) and (c) can be handled similarly.

We'd like to compare $t-G_{rr}(t)$, the amount of sticking for Y_r^τ and $Y_r^{\tau'}$, to $t-C(t)$, the amount of sticking for S_0^τ and $S_0^{\tau'}$. We'll accomplish this by constructing coupled versions of the two processes. In both cases there are two paths that alternate between identical sticking sections and independent non-sticking sections. To be specific, we take $T_0 = T_0^* := 0$ and for $k \geq 0$ define:

$$\begin{aligned}
T_{2k+1} :=& \inf\{k \geq T_{2k} : \text{ The clock at } S_0^\tau(k) = S_0^{\tau'}(k) \text{ rings in } [\tau, \tau']\},\\
T_{2k+2} :=& \inf\{k > T_{2k+1} : \ S_0^\tau(k) = S_0^{\tau'}(k)\},\\
\Delta_k :=& T_{2k+1} - T_{2k} \geq 0, \ \Gamma_k := T_{2k+2} - T_{2k+1} \geq 1,
\end{aligned}$$

and:

$$\begin{aligned}
T_{2k+1}^* :=& \inf\{k \geq T_{2k} : \ k \text{ is not an } rr\text{-sticking time}\},\\
T_{2k+2}^* :=& \inf\{k > T_{2k+1} : \ Y_r^\tau(k) = Y_r^{\tau'}(k)\},\\
\Delta_k^* :=& T_{2k+1}^* - T_{2k}^* \geq 0, \ \Gamma_k^* := T_{2k+2}^* - T_{2k+1}^* \geq 1.
\end{aligned}$$

Then on $[T_{2k}^{(*)}, T_{2k+1}^{(*)}]$ we have S_0^τ and $S_0^{\tau'}$ (Y_r^τ and $Y_r^{\tau'}$) sticking for $\Delta_k^{(*)}$ steps, while on $[T_{2k+1}^{(*)}, T_{2k+2}^{(*)}]$ they move independently until meeting at $T_{2k+2}^{(*)}$. Notice that Γ_k and Γ_k^* have the same distribution, they are both excursion times for pairs of independent random walks. So we may take $\Gamma_k = \Gamma_k^*$ for our coupled versions. To compare Δ_k, Δ_k^*, notice that:

$$\mathbb{P}\left(\Delta_k^{(*)} \geq j\right) = \prod_{i=1}^{j} \mathbb{P}\left(\Delta_k^{(*)} \geq i | \Delta_k^{(*)} \geq i-1\right)$$

and:

$$\mathbb{P}\left(\Delta_k^* \geq i | \Delta_k^* \geq i-1\right) \leq \mathbb{P}\left(\Delta_k \geq i | \Delta_k \geq i-1\right) \text{ for all } i \geq 1, \tag{10}$$

so:

$$\mathbb{P}\left(\Delta_k^* \geq j\right) \leq \mathbb{P}\left(\Delta_k \geq j\right) \text{ for all } j, k \geq 0. \tag{11}$$

To see (10), consider that $\mathbb{P}\left(\Delta_k \geq i | \Delta_k \geq i-1\right)$ is just the probability of no clock ring in $[\tau, \tau']$. For Δ_k^*, we have the probability that $Y_r^\tau = Y_r^{\tau'} \neq Y_l^\tau, Y_l^{\tau'}$ and there is no $\mathcal{W}$-ring, or $Y_r^\tau = Y_r^{\tau'} = Y_l^\tau = Y_l^{\tau'}$ and there is no $\hat{\mathcal{W}}$-ring. These are disjoint events and the clocks are independent of the positions of previous arrows, so this is bounded by the probability of no clock ring.

Combining this with the above observations, we can couple Δ_k, Δ_k^* and Γ_k, Γ_k^* such that $\Delta_k^* \leq \Delta_k$ and $\Gamma_k = \Gamma_k^*$. This means that the rr-sticking sections are shorter than the $S_0^\tau, S_0^{\tau'}$ sticking sections, while the independent sections have the same length. This implies $t - G_{rr}(t) \leq t - C(t)$ for the coupled versions, which shows (d) is stochastically bounded by $t - C(t)$. This can be proven for $(b), (c)$ by a nearly identical coupling argument, where the portion of the left/right paths after their first meeting is coupled with $S_0^\tau, S_0^{\tau'}$. So we've shown that $(a), (b), (c), (d)$ are each stochastically bounded by $t - C(t)$. Combining this with (9) we get:

$$\begin{aligned}
\mathbb{P}\left(\sup_{t\in[0,1]}(t-\bar{G}(t)) \geq \Delta^\beta\right) =& \mathbb{P}\left(\sup_{t\in[0,d_k^2]}(t-G(t)) \geq d_k^2\Delta^\beta\right) \\
\leq& 4\mathbb{P}\left(\sup_{t\in[0,d_k^2]}(t-C(t)) \geq d_k^2\frac{\Delta^\beta}{4}\right) \text{ (using (9) and above paragraph)} \\
=& 4\mathbb{P}\left(\sup_{t\in[0,1]}(t-\bar{C}(t)) \geq \frac{\Delta^\beta}{4}\right) \\
\leq& 4\tilde{c}\left(\frac{\Delta}{4^{1/\beta}}\right)^{1-\beta} \text{ (by Lemma 3.4 from [2])} \\
=& c''\,\Delta^{1-\beta}.
\end{aligned}$$

This completes the proof since $\tilde{c}$, and thus c'', is independent of k, τ and τ'. $\square$

Now we define C_d^τ to be the rectangle event for $Y_{l_d}^\tau, Y_{r_d}^\tau$. That is:

$$\begin{aligned}
C_d^\tau :=& \{|Y_{l_d}^\tau(t)| \leq d_k \text{ and } |Y_{r_d}^\tau(t)| \leq d_k \;\; \forall t \in [0, d_k^2]\} \\
=& \{|\tilde{Y}_{l_d}^\tau(t)| \leq 1 \text{ and } |\tilde{Y}_{r_d}^\tau(t)| \leq 1 \;\; \forall t \in [0,1]\}.
\end{aligned}$$

Given $r > 0$ we define the r-approximations of our rectangle events as:

$$\begin{aligned}
\{C_{(d)}^\tau + r\} :=& \{|Y_{l_{(d)}}^\tau(t)| \leq (1+r)d_k \text{ and } |Y_{r_{(d)}}^\tau(t)| \leq (1+r)d_k \;\; \forall t \in [0, d_k^2]\} \\
=& \{|\tilde{Y}_{l_{(d)}}^\tau(t)| \leq 1+r \text{ and } |\tilde{Y}_{r_{(d)}}^\tau(t)| \leq 1+r \;\; \forall t \in [0,1]\}.
\end{aligned}$$

Recall that $Y_{l_d}^\tau, Y_{r_d}^\tau$ are independent of $Y_{l_d}^{\tau'}, Y_{r_d}^{\tau'}$, and therefore:

$$C_d^\tau(\{C_d^\tau + r\}) \text{ is independent of } C_d^{\tau'}(\{C_d^{\tau'} + r\}). \tag{12}$$

We also have:

$$(Y_{l_d}^\tau, Y_{r_d}^\tau) \stackrel{d}{=} (Y_l^\tau, Y_r^\tau), \tag{13}$$

since both are just pairs of independent random walks. So:

$$\mathbb{P}(C_d^\tau) = \mathbb{P}(C^\tau) = \mathbb{P}(C^0). \tag{14}$$

We will need the following adaptation of Lemma 3.3 from [2]:

Lemma 2. *Given any $\alpha < 1/2$, there is $c' \in (0,\infty)$ independent of Δ, k such that:*

$$\mathbb{P}\left(\{C_d^\tau + \Delta^\alpha\} \setminus C_d^\tau\right) \le c'\Delta^\alpha.$$

Proof.

$$\mathbb{P}\left(\{C_d^\tau + \Delta^\alpha\} \setminus C_d^\tau\right) \le \mathbb{P}\left(\inf_{t\in[0,1]} \tilde{Y}^\tau_{l_d}(t) \in [-1-\Delta^\alpha, -1)\right) + \mathbb{P}\left(\sup_{t\in[0,1]} \tilde{Y}^\tau_{l_d}(t) \in (1, 1+\Delta^\alpha]\right)$$
$$+\mathbb{P}\left(\inf_{t\in[0,1]} \tilde{Y}^\tau_{r_d}(t) \in [-1-\Delta^\alpha, -1)\right) + \mathbb{P}\left(\sup_{t\in[0,1]} \tilde{Y}^\tau_{r_d}(t) \in (1, 1+\Delta^\alpha]\right).$$

Now each of the four terms on the right is bounded by $c\,\Delta^\alpha$. This follows exactly as in the proof of Lemma 3.3 in [2]. To see this, note that the $\tilde{Y}$'s are simple symmetric random walks started at $\pm d_{k-1}/d_k \in [-1,1]$, diffusively rescaled by $\delta = d_k^{-1}$. We can thus approximate the $\tilde{Y}$'s by Brownian motion paths (for details see [3] and [2], Lemma 3.3). The result then follows, as the maximum (minimum) process of a Brownian motion has a bounded distribution function. □

The final ingredient for the proof of Proposition 1 is a bound on the modulus of continuity of a random walk. This is given by Lemma 3.5 from [2]:

Lemma 3. *(Lemma 3.5, [2])*
Let $S(t)$ be a simple symmetric random walk and define $\tilde{S}(t) := S(t/\delta^2)\delta$.
Let $\omega_{\tilde{S}}(\epsilon) := \sup_{s,t\in[0,1],|s-t|<\epsilon} |\tilde{S}(t) - \tilde{S}(s)|$ be the modulus of continuity of $\tilde{S}$. Let $\alpha, \beta \in (0,\infty)$ be such that $\beta/2 > \alpha$. For any $r \ge 0$, there exists c (independent of Δ and δ) such that:

$$\mathbb{P}\left(\omega_{\tilde{S}}(\Delta^\beta) \ge \frac{\Delta^\alpha}{2}\right) \le c\,\Delta^r.$$

This is a consequence of the Garsia-Rodemich-Rumsey inequality [4]. For a proof see [2].

We may now prove Proposition 1. The remaining steps are nearly identical to the proof of Proposition 3.1 from [2] (see the end of Section 3). We include them for the sake of completeness.

For any $0 < \alpha < 1/2$, we have:

$$\begin{aligned}\mathbb{P}\left(C^\tau \cap C^{\tau'}\right) \le& \mathbb{P}\left(\{C_d^\tau + \Delta^\alpha\} \cap \{C_d^{\tau'} + \Delta^\alpha\}\right) \\ &+2\mathbb{P}\left(C^\tau \setminus \{C_d^\tau + \Delta^\alpha\}\right),\end{aligned} \tag{15}$$

where we used the equidistribution of $(C^\tau, \{C_d^\tau + \Delta^\alpha\})$ and $(C^{\tau'}, \{C_d^{\tau'} + \Delta^\alpha\})$. Using (12)-(14) we get:

$$\begin{aligned}\mathbb{P}(\{C_d^\tau + \Delta^\alpha\} \cap \{C_d^{\tau'} + \Delta^\alpha\}) =& \mathbb{P}(\{C_d^\tau + \Delta^\alpha\})\mathbb{P}(\{C_d^{\tau'} + \Delta^\alpha\}) \\ \le& \mathbb{P}(C_d^\tau)^2 + 2\mathbb{P}(\{C_d^\tau + \Delta^\alpha\} \setminus C_d^\tau) \\ =& \mathbb{P}(C^0)^2 + 2\mathbb{P}(\{C_d^\tau + \Delta^\alpha\} \setminus C_d^\tau).\end{aligned} \tag{16}$$

Combined with Lemma 2 this gives:

$$\mathbb{P}(\{C_d^\tau + \Delta^\alpha\} \cap \{C_d^{\tau'} + \Delta^\alpha\}) \le \mathbb{P}(C^0)^2 + 2c'\Delta^\alpha. \tag{17}$$

Now that we have (15) and (17) we just need $\hat{c}, a'$ such that:

$$\mathbb{P}\left(C^\tau \setminus \{C_d^\tau + \Delta^\alpha\}\right) \leq \hat{c}\,\Delta^{a'}. \tag{18}$$

Recall the splitting of the Y^τ's given by (8). Analogous considerations for the $\tilde{Y}^\tau$'s gives:

$$\begin{aligned}
\tilde{Y}_l^\tau(t) =& \tilde{Y}_{l_d}^\tau(\bar{G}(t)) + \tilde{Y}_{l_s}^\tau(t - \bar{G}(t)) \\
=& \tilde{Y}_{l_d}^\tau(t) + [\tilde{Y}_{l_d}^\tau(\bar{G}(t)) - \tilde{Y}_{l_d}^\tau(t)] + \tilde{Y}_{l_s}^\tau(t - \bar{G}(t)), && (19)\\
\tilde{Y}_r^\tau(t) =& \tilde{Y}_{r_d}^\tau(t) + [\tilde{Y}_{r_d}^\tau(\bar{G}(t)) - \tilde{Y}_{r_d}^\tau(t)] + \tilde{Y}_{r_s}^\tau(t - \bar{G}(t)). && (20)
\end{aligned}$$

Notice that all the $\tilde{Y}$'s appearing in (19), (20) are simple symmetric random walks rescaled by $\delta = d_k^{-1}$, as in Lemma 3. Also, we've taken $\alpha < 1/2$, so we may choose $0 < \beta < 1$ such that $\beta/2 > \alpha$. Then:

$$\begin{aligned}
\mathbb{P}\left(C^\tau \setminus \{C_d^\tau + \Delta^\alpha\}\right) \leq& \mathbb{P}\left(|\tilde{Y}_l^\tau - \tilde{Y}_{l_d}^\tau|_\infty \geq \Delta^\alpha\right) + \mathbb{P}\left(|\tilde{Y}_r^\tau - \tilde{Y}_{r_d}^\tau|_\infty \geq \Delta^\alpha\right) \\
\leq& \mathbb{P}\left(|\tilde{Y}_{l_d}^\tau(\bar{G}(t)) - \tilde{Y}_{l_d}^\tau(t)|_\infty \geq \frac{\Delta^\alpha}{2}\right) + \mathbb{P}\left(|\tilde{Y}_{l_s}^\tau(t - \bar{G}(t))|_\infty \geq \frac{\Delta^\alpha}{2}\right) \\
&+ \mathbb{P}\left(|\tilde{Y}_{r_d}^\tau(\bar{G}(t)) - \tilde{Y}_{r_d}^\tau(t)|_\infty \geq \frac{\Delta^\alpha}{2}\right) + \mathbb{P}\left(|\tilde{Y}_{r_s}^\tau(t - \bar{G}(t))|_\infty \geq \frac{\Delta^\alpha}{2}\right) \\
\leq& 4\mathbb{P}\left(\omega_{\tilde{S}}(\Delta^\beta) \geq \frac{\Delta^\alpha}{2}\right) + 4\mathbb{P}\left(\sup_{t\in[0,1]}(t - \bar{G}(t)) \geq \Delta^\beta\right) \\
\leq& 4c\,\Delta^r + 4c''\,\Delta^{1-\beta},
\end{aligned}$$

where $|\cdot|_\infty$ denotes the sup norm restricted to $[0,1]$. The last inequality follows from Lemmas 1 and 3. This completes the proof of Proposition 1.

4 Proof of Theorem 1

Now that we have Proposition 1 we are almost ready to prove Theorem 1. We'd like to show the existence of exceptional times at which $\cap_{k\geq 0} C_k^\tau$ occurs. We just need one more Lemma from [2]:

Lemma 4. *(Lemma 4.3, [2]) There exists $c \in (0,\infty)$ such that for $\tau, \tau' \in [0,1]$, $\forall n \geq 0$:*

$$\prod_{k=0}^{n} \frac{\mathbb{P}(C_k^\tau \cap C_k^{\tau'})}{\mathbb{P}(C_k)^2} \leq c\frac{1}{|\tau - \tau'|^b},$$

where $C_k := C_k^0$ and $b = \log(\sup_k[\mathbb{P}(C_k)^{-1}])/\log\gamma > 0$.

This was established in [2] for a different collection of rectangle events, A_k. However, to make their proof work for C_k, we just need a, c such that:

$$\mathbb{P}(C_k^\tau \cap C_k^{\tau'}) \leq \mathbb{P}(C_k)^2 + c\left(\frac{1}{\gamma^k|\tau - \tau'|}\right)^a \quad \forall \tau, \tau' \in [0,1], k \geq 0, \tag{21}$$

and:

$$\sup_k [\mathbb{P}(C_k)^{-1}]) < \infty. \tag{22}$$

(21) follows from Proposition 1. To see (22), notice that the rectangles R_k grow diffusively, and therefore $\mathbb{P}(C_k) \longrightarrow \mathbb{P}(C_\infty)$, the probability of the corresponding rectangle event for Brownian motion paths. So Lemma 4 follows exactly as in [2], see [2] for the details.

Theorem 1 now follows as in [2],[10]. We will repeat their arguments for the sake of completeness. The Cauchy-Schwartz inequality and Lemma 4 give, $\forall n \geq 0$:

$$\mathbb{P}\left(\int_0^1 \prod_{k=0}^n \mathbb{1}_{C_k^\tau} d\tau > 0\right) \geq \frac{\left(\mathbb{E}\left[\int_0^1 \prod_{k=0}^n \mathbb{1}_{C_k^\tau} d\tau\right]\right)^2}{\mathbb{E}\left[\left(\int_0^1 \prod_{k=0}^n \mathbb{1}_{C_k^\tau} d\tau\right)^2\right]} \tag{23}$$

$$= \left[\int_0^1 \int_0^1 \prod_{k=0}^n \frac{\mathbb{P}(C_k^\tau \cap C_k^{\tau'})}{\mathbb{P}(C_k)^2} d\tau d\tau'\right]^{-1} \tag{24}$$

$$\geq c^{-1}\left[\int_0^1 \int_0^1 \frac{1}{|\tau - \tau'|^b} d\tau d\tau'\right]^{-1}, \tag{25}$$

where (24) comes from the independence of the arrow configurations in different R_k's and the stationarity of $\tau \longrightarrow \mathcal{W}(\tau)$. We would like to show that (25) is strictly positive. Lemma 4 gave:

$$b = \log(\sup_k [\mathbb{P}(C_k)^{-1}]) / \log \gamma.$$

Recall that the R_k's, and thus the $\mathbb{P}(C_k)$'s, depend on γ. As γ increases, R_0 remains the same, while for $k \geq 1$, R_k grows diffusively. The size of R_{k-1} relative to R_k also tends to zero, so the starting points of $X_{l_k}^\tau, X_{r_k}^\tau$ converge to the center of the rectangle when diffusively rescaled. This implies that as γ goes to infinity, $\sup_k[\mathbb{P}(C_k)^{-1}]$ converges to $\max\{\mathbb{P}(C_0)^{-1}, \mathbb{P}(C^*)^{-1}\}$, where C^* is the rectangle event for two independent Brownian motions started in the center. So for γ sufficiently large we have $b < 1$, and thus $|\tau - \tau'|^{-b}$ integrable on $[0,1] \times [0,1]$. (23)-(25) then imply:

$$\inf_n \mathbb{P}\left(\int_0^1 \prod_{k=0}^n \mathbb{1}_{C_k^\tau} d\tau > 0\right) \geq p > 0. \tag{26}$$

Letting $E_n := \{\tau \in [0,1] : \cap_{k=0}^n C_k^\tau \text{ occurs}\}$, (26) then implies $\mathbb{P}(\cap_{n=0}^\infty \{E_n \neq \emptyset\}) \geq p > 0$. Notice that the E_n are decreasing in n. So if the E_n were closed, this would imply $\mathbb{P}((\cap_{n=0}^\infty E_n) \neq \emptyset) \geq p > 0$ and (7), and thus Theorem 1, would follow.

Unfortunately, the E_n are not closed. This is handled as in [2],[6](Lemma 3.2) by noting that the E_n are nested collections of intervals, and their endpoints must be switching times for some arrow in $\mathcal{W}$ or $\hat{\mathcal{W}}$. There are only countably many switching times, and the locations of an arrow's switching times are independent of the configuration of the rest of the arrows. This means that $\mathcal{W}(\tau)$ and $\hat{\mathcal{W}}(\tau)$ are distributed as discrete webs for all switching times, τ. This will imply that, almost surely:

$$\bigcap_{n=0}^\infty E_n = \bigcap_{n=0}^\infty \bar{E}_n. \tag{27}$$

To see this, assume (27) is not true, i.e., that with positive probability there exists a τ that is in the right set but not the left. Then for some m, τ is in $\bar{E}_m$, but not E_m. This implies that τ is the right endpoint to an interval from E_m (by right continuity), and thus must be a switching time for exactly one arrow, ξ_*^τ, from $\mathcal{W}$ or $\hat{\mathcal{W}}$. Since the E_n are nested and τ is in $\bar{E}_n$ for all n, τ must also be a right endpoint to an interval from E_k, for all $k \geq m$. This means that for all n, either $\cap_{k=0}^n C_k^\tau$ occurs, or there is an $\epsilon > 0$ such that $\cap_{k=0}^n C_k^{\tau'}$ occurs for $\tau' \in [\tau - \epsilon, \tau)$. In the second case, $\cap_{k=0}^n C_k^\tau$ ceases to occur only due to the resetting of ξ_*^τ (since no other arrow could switch at the same time). So if we take the webs at time τ, and we reset this arrow to its previous position, then the above discussion implies that $\cap_{k=0}^\infty C_k^\tau$ will occur. But $\mathcal{W}(\tau)$ and $\hat{\mathcal{W}}(\tau)$ are distributed as discrete webs, so switching one arrow cannot cause a probability zero event to happen. This is a contradiction, and it came from the assumption that (27) was not true. So we've proven (27) and Theorem 1 then follows from the discussion in the previous paragraph.

5 Hausdorff Dimensions of Sets of Exceptional Times

In this section we look at the sets of exceptional times and examine their Hausdorff dimensions. We've shown the existence of exceptional times at which $|S_0^\tau(t)|$ remains bounded by $j + K\sqrt{t}$ and exceptional times at which $\limsup_{t\to\infty} |S_0^\tau(t)|/\sqrt{t} \leq K$. The strategy was to show that S_0^τ remained within a stack of diffusively growing rectangles. The size of the rectangles, and thus the values of K in our bounds, was determined by a parameter, γ. This next proposition attempts to capture these relationships between K and γ.

Proposition 2. *Let $\sigma_\gamma(t)$ denote the right edge of $\cup_{k\geq 0} R_k(\gamma)$. We have:*

$$\sigma_\gamma(t) \leq 2 + \gamma\sqrt{t} \quad \text{for all } t \geq 0, \tag{28}$$

$$\limsup_{t\to\infty} \frac{\sigma_\gamma(t)}{\sqrt{t}} \leq \sqrt{\gamma^2 - 1}. \tag{29}$$

Proof. Recall that $d_k = 2(\lfloor \frac{\gamma^k}{2} \rfloor + 1)$, so $\gamma^k \leq d_k \leq \gamma^k + 2$. This gives:

$$t_k = d_0^2 + d_1^2 + \ldots + d_{k-1}^2 \geq \gamma^0 + \gamma^2 + \ldots \gamma^{2(k-1)} = \frac{\gamma^{2k} - 1}{\gamma^2 - 1}.$$

Now, for $t_k \leq t < t_{k+1}$, $k \geq 1$ we have:

$$\begin{aligned} \sigma_\gamma(t) =& d_k \leq \gamma^k + 2 \\ \leq& \gamma^k \left(\frac{\gamma^{2k} - 1}{\gamma^2 - 1}\right)^{-\frac{1}{2}} \sqrt{t} + 2 = \sqrt{\frac{\gamma^2 - 1}{1 - \gamma^{-2k}}}\sqrt{t} + 2, \end{aligned} \tag{30}$$

and (29) follows immediately. To see (28), notice that for $t < t_1$ we have $\sigma_\gamma(t) = d_0 = 2 \leq 2 + \gamma\sqrt{t}$. For $t \geq t_1$, we see $\sqrt{(\gamma^2 - 1)/(1 - \gamma^{-2k})} \leq \gamma$ when $k \geq 1$, so (28) follows from (30). □

Now we'd like to consider various sets of exceptional times. For non-negative $j \in \mathbb{Z}$, we define:

$$\begin{aligned} T_j^\pm(K) :=& \{\tau \in [0,\infty) : |S_0^\tau(t)| \leq j + K\sqrt{t} \quad \forall t\}, \; T_\infty^\pm(K) := \cup_{j\geq 0} T_j^\pm(K), \\ T_j^+(K) :=& \{\tau \in [0,\infty) : S_0^\tau(t) \leq j + K\sqrt{t} \quad \forall t\}, \quad T_\infty^+(K) := \cup_{j\geq 0} T_j^+(K), \\ T_j^-(K) :=& \{\tau \in [0,\infty) : S_0^\tau(t) \geq -j - K\sqrt{t} \quad \forall t\}, \; T_\infty^-(K) := \cup_{j\geq 0} T_j^-(K). \end{aligned}$$

We are interested in the Hausdorff dimensions of these sets. As in [2],[6], ergodicity of the DyDW in τ implies the dimension of any set of exceptional times will be almost surely constant. Future discussions of such dimensions should be understood to refer to this constant, and thus may only hold almost surely. In [2](Proposition 5.2) it was shown that in the one-sided case, the Hausdorff dimensions of $T_j^+(K)$ and $T_j^-(K)$ do not depend on $j \geq 0$. So:

$$\dim_H(T_0^+(K)) = \dim_H(T_\infty^+(K))(= \dim_H(T_j^+(K)) \text{ for all } j),$$
$$\dim_H(T_0^-(K)) = \dim_H(T_\infty^-(K))(= \dim_H(T_j^-(K)) \text{ for all } j).$$

Modifying their argument, we obtain:

Proposition 3. *For $T_j^\pm(K)$ as defined above, we have:*

$$\sup_{K'<K} dim_H(T_\infty^\pm(K')) \leq \dim_H(T_1^\pm(K)) \leq \dim_H(T_\infty^\pm(K)) \leq \inf_{K''>K} \dim_H(T_1^\pm(K'')).$$

The reason we take $j = 1$ instead of $j = 0$ is to prevent the first step of the walk from pushing $|S_0^\tau|$ past $j + K\sqrt{t}$ when $K < 1$. If we are only interested in $K \geq 1$ we can take $j = 0$ and obtain analogous bounds involving $T_0^\pm(K)$. Notice that $T_1^\pm(K)$ and $T_\infty^\pm(K)$ are increasing functions of K, and thus must be continuous for all but countably many K. So for all but countably many K, the inequalities from Proposition 3 collapse into equalities, and the dimensions will not depend on j. It would seem natural that at least the center inequality should be an equality for all K, giving j-independence as in the one-sided case, but we are unable to prove this. The proof of Proposition 3 is motivated by the proof of Proposition 5.2 from [2]. The second inequality is trivial; the first and third follow from the same argument. To see this, pick any $K_1 < K_2, j \geq 1$ and notice that:

$$\{\tau : |S_0^\tau(t)| \leq 1 \text{ for all } t \in [0, 2n]\} \cap \{\tau : |S_{(0,2n)}^\tau(t)| \leq j + K_1\sqrt{t - 2n} \text{ for all } t \geq 2n\}$$

is contained in $T_1^\pm(K_2)$ for n sufficiently large. This is because $S_0^\tau(2n) = 0$ on the first set, and $j + K_1\sqrt{t - 2n} \leq 1 + K_2\sqrt{t}$ for large n (this fails for $K_1 = K_2$, which is why we don't get full j-independence). The second set is just a translated version of $T_j^\pm(K_1)$, and thus has the same Hausdorff dimension. The first set consists of τ at which an independent (of $S_{(0,2n)}^\tau$) event of positive probability occurs. Thus, by the same ergodicity arguments used in Proposition 5.2 from [2], intersection with the first set does not decrease the dimension. So:

$$\dim_H(T_j^\pm(K_1)) \leq \dim_H(T_1^\pm(K_2)) \text{ for all } j \geq 1, K_1 < K_2,$$

which proves both the first and third inequalities.

Now we focus on comparing the Hausdorff dimensions of $T_\infty^\pm(K)$ and other, related sets of exceptional times. We may drop j from the notation, and when j is not specified it should be understood that we are discussing $T_\infty^\pm(K)$ (i.e., $T^\pm(K) := T_\infty^\pm(K)$). Proposition 3 allows us to translate the coming bounds into bounds for $\dim_H(T_1^\pm(K))$ (or $\dim_H(T_0^\pm(K))$ for $K \geq 1$).

We now consider dynamical times at which $S_0^\tau(t)$ displays exceptional behaviour as t goes to ∞. That is, we look at times at which S_0^τ *is K-subdiffusive eventually:*

$$\hat{T}^\pm(K) := \{\tau \in [0, \infty) : \exists N, j \text{ s.t. } |S_0^\tau(t)| \leq j + K\sqrt{t} \quad \forall t \geq N\}$$

and times at which S_0^τ *is K-subdiffusive in the limit:*

$$\tilde{T}^\pm(K) := \{\tau \in [0, \infty) : \limsup_{t\to\infty} |S_0^\tau(t)|/\sqrt{t} \leq K\}.$$

Notice that:

$$\tilde{T}^{\pm}(K) = \bigcap_{K'>K} \hat{T}^{\pm}(K'),$$

so:

$$\dim_H(\tilde{T}^{\pm}(K)) = \inf_{K'>K} \dim_H(\hat{T}^{\pm}(K')). \tag{31}$$

As in the discussion following Proposition 3, monotonicity in K implies that (31) will also equal $\dim_H(\hat{T}^{\pm}(K))$ except for at most countably many K. We now compare these sets with $T^{\pm}(K)$:

Proposition 4. *For $\hat{T}^{\pm}(K)$, $T^{\pm}(K)(= T^{\pm}_{\infty}(K))$ as defined above, we have:*

$$\dim_H(\hat{T}^{\pm}(K)) = \dim_H(T^{\pm}(K)). \tag{32}$$

Proof. Almost surely, for all τ, all walks in the DyDW are recurrent, and all pairs of walks coalesce (see Theorem 2.1, Remark 2.3 from [2]). This implies:

$$\begin{aligned}\hat{T}^{\pm}(K) &= \{\tau \in [0,\infty) : \exists N, j \text{ s.t. } |S^{\tau}_0(t)| \le j + K\sqrt{t} \quad \forall t \ge N\} \\ &\overset{\text{a.s.}}{=} \bigcup_{n\ge 0} \{\tau \in [0,\infty) : \exists j \text{ s.t. } |S^{\tau}_{(0,2n)}(t)| \le j + K\sqrt{t-2n} \quad \forall t \ge 2n\}.\end{aligned} \tag{33}$$

To see this, notice that on the second set, S^{τ}_0 will a.s. eventually coalesce with $S^{\tau}_{(0,2n)}$, so for t large we will have $|S^{\tau}_0(t)| = |S^{\tau}_{(0,2n)}(t)| \le j + K\sqrt{t-2n} \le j + K\sqrt{t}$. For τ in the first set, let N^* be the first time S^{τ}_0 returns to zero after N. Then $|S_{(0,N^*)}(t)| \le j + K\sqrt{t} \le (j + K\sqrt{N^*}) + K\sqrt{t-N^*}$. This proves (33). To complete the proof, just notice that the second set is a countable union of translated versions of $T^{\pm}(K)$, and thus has the same Hausdorff dimension. $\square$

Using (33) and the recurrence of all paths, we also get:

$$\hat{T}^{\pm}(K) \overset{\text{a.s.}}{=} \bigcap_{m\ge 0} \bigcup_{n\ge m} \{\tau \in [0,\infty) : \exists j \text{ s.t. } |S^{\tau}_{(0,2n)}(t)| \le j + K\sqrt{t-2n} \quad \forall t \ge 2n\},$$

which is a tail random variable with respect to the underlying $\xi^{\tau}_{(x,t)}$ processes. Similar reasoning also applies to $\tilde{T}^{\pm}(K)$. These observations imply:

$$\begin{aligned}&\mathbb{P}(\hat{T}^{\pm}(K) \cap [0,\epsilon] = \emptyset) = 0 \text{ or } 1 \qquad \text{for all } K > 0, \epsilon \ge 0, \\ &\mathbb{P}(\tilde{T}^{\pm}(K) \cap [0,\epsilon] = \emptyset) = 0 \text{ or } 1 \qquad \text{for all } K > 0, \epsilon \ge 0.\end{aligned} \tag{34}$$

An easy consequence of (34) is given by the following proposition:

Proposition 5. *Almost surely, for every $K > 0$, $\tilde{T}^{\pm}(K)$, $\hat{T}^{\pm}(K)$ will each be either empty, or dense in $[0,\infty)$.*

Now we prove a lower bound for the Hausdorff dimension of $\tilde{T}^{\pm}(K)$. The above results (Proposition 3, (31), Proposition 4) allow us to translate this into lower bounds for $\dim_H(T^{\pm}(K))$, $\dim_H(T^{\pm}_j(K))$ and $\dim_H(\hat{T}^{\pm}(K))$. In fact, our lower bound is continuous in K, so we get the same bound for all these sets of exceptional times. Now, let $\tilde{\gamma}(K) := \sqrt{K^2+1}$, so that:

$$\limsup_{t\to\infty} \frac{\sigma_{\tilde{\gamma}(K)}(t)}{\sqrt{t}} \le K \tag{35}$$

(see Proposition 2). Given γ, let $C_\infty(\gamma)$ be the corresponding rectangle event for Brownian motions. That is, the event that two independent Brownian motions started at $\pm\gamma^{-1}$ stay within $[-1,1]$ for $0 \le t \le 1$. Then we have:

Proposition 6.

$$\dim_H(\tilde{T}^{\pm}(K)) \geq 1 - \frac{\log \mathbb{P}(C_\infty(\tilde{\gamma}(K)))^{-1}}{\log \tilde{\gamma}(K)} =: 1 - b_\infty(K). \tag{36}$$

This is established by a modification of the arguments used in Proposition 5.3 from [2]. As in [2], we drop K from the notation. First we define a family of random measures, $\sigma_{n,m}$, that play the role of the σ_n from their proof. As above, we take $C_k := C_k^0$. Given a Borel set E in $[0,1]$, $n \geq m$, we define:

$$\sigma_{n,m}(E) := \int_E \prod_{k=m}^{n} \frac{\mathbb{1}_{C_k^\tau}}{\mathbb{P}(C_k)} d\tau,$$

and notice that $\sigma_{n,m}$ is supported on $\bar{E}_{n,m}$, the closure of:

$$E_{n,m} := \{\tau \in [0,1] : \cap_{k=m}^{n} C_k^\tau \text{ occurs}\}. \tag{37}$$

Now, reasoning as in Lemma 4.3 from [2], we have for all $n \geq m$:

$$\begin{aligned}\mathbb{E}[\sigma_{n,m}([0,1])^2] &= \int_0^1 \int_0^1 \prod_{k=m}^{n} \frac{\mathbb{P}(C_k^\tau \cap C_k^{\tau'})}{\mathbb{P}(C_k)^2} d\tau d\tau' \\ &\leq c \int_0^1 \int_0^1 \frac{1}{|\tau - \tau|^{b_m}} d\tau d\tau',\end{aligned}$$

with:

$$b_m = \log[\sup_{k \geq m}(\mathbb{P}(C_k)^{-1})] / \log \tilde{\gamma}.$$

Now, we assume that K is large enough to make the right-hand side of (36) positive (otherwise there is nothing to prove). Then, by the diffusive scaling of the events C_k, $b_m \to b_\infty < 1$ as $m \to \infty$. So given any $\alpha < 1 - b_\infty$, there exists m large enough such that $\alpha + b_m < 1$. Now, arguing as in [2], we may use the extension of Frostman's lemma from [10] to conclude that:

$$\dim_H(\cap_{n \geq m} \bar{E}_{n,m}) \geq \alpha \text{ with positive probability.} \tag{38}$$

Now, for our chosen K, $\cap_{n \geq m} E_{n,m} \subset \tilde{T}^{\pm}(K)$ for all m (using (35) and the a.s. coalescence of all paths). We've shown that given any $\alpha < 1 - b_\infty$, (38) holds for some sufficiently large m. Also, $\cap_{n \geq m} E_{n,m} = \cap_{n \geq m} \bar{E}_{n,m}$ almost surely, by the same argument used to establish (27). Combining these observations, we have:

$$\dim_H(\tilde{T}^{\pm}(K)) \geq 1 - b_\infty(K),$$

(almost surely by ergodicity in τ of the DyDW). This proves Proposition 6.

Remark 2. One may obtain an analogous lower bound for the one-sided sets of exceptional times by considering one-sided versions of our rectangle events (i.e. $\{S^{\tau}_{l_k}(t) \geq -d_k \ \ \forall t \in [t_k, t_{k+1}]\}$). This lower bound should be slightly better than the one given in [2]. One may also wish to consider "asymmetrical" exceptional times. That is, exceptional times where the K of $\tilde{T}^{\pm}(K)$, $\hat{T}^{\pm}(K)$, $T^{\pm}(K)$, $T^{\pm}_j(K)$, etc. is replaced by two constants, K_L, and K_R, giving different bounds on the left and right sides. One can obtain an analogous lower bound for the dimension of these asymmetrical exceptional times using the "skewed rectangle" construction described in Remark 1.

Now we look at upper bounds for the Hausdorff dimension of the sets of two-sided exceptional times. This is a straightforward extension of the results in Section 5.2 of [2]. Following [2], we state the results for the asymmetrical case. So we give an upper bound for $\dim_H(T^-_1(K_L) \cap T^+_1(K_R))$. Recall:

$$T^-_1(K_L) \cap T^+_1(K_R) = \{\tau \in [0, \infty) : -1 - K_L\sqrt{t} \leq S^{\tau}_0(t) \leq 1 + K_R\sqrt{t} \ \text{ for all } t\}, \tag{39}$$

using the definitions given earlier in this section.

In [2], Proposition 5.5 they prove the bound $\dim_H(T^-_1(K)) \leq 1 - p(K)$, where $p(K) \in (0, 1)$ is the solution to:

$$f(p, K) := \frac{\sin(\pi p/2)\Gamma(1 + p/2)}{\pi} \sum_{n=1}^{\infty} \frac{(\sqrt{2}K)^n}{n!} \Gamma((n - p)/2) = 1. \tag{40}$$

They also prove that $T^-_1(K_L) \cap T^+_1(K_R)$ is empty when $p(K_L) + p(K_R) > 1$ (see [2], Proposition 5.8). The function $p(K)$ comes from [8], where it is shown that $p(K)$ is continuous and decreasing on $(0, \infty)$, tending to 0 as K goes ∞, tending to 1 as K goes to 0.

The upper bound from [2] is established by partitioning $[0, 1]$ into intervals of equal length, and estimating the number of these needed to cover $T^-_1(K)$. An application of the FKG inequality, as in Proposition 5.8 of [2], extends the bound to the two-sided case, giving:

Proposition 7.

$$\dim_H(T^-_1(K_L) \cap T^+_1(K_R)) \leq 1 - p(K_L) - p(K_R),$$

so:

$$\dim_H(T^{\pm}_1(K)) \leq 1 - 2p(K).$$

Note that, as with the lower bound, continuity of the bound combined with our previous results gives an identical bound for $\dim_H(T^{\pm}(K))$, $\dim_H(T^{\pm}_j(K))$, $\dim_H(\tilde{T}^{\pm}(K))$, $\dim_H(\hat{T}^{\pm}(K))$, and their asymmetrical analogues.

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